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Discrete evolution for the zero modes of the quantum Liouville model

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Abstract
The dynamical system for the zero modes of the Liouville model, which is separated from the full dynamics for the discrete shifts of time $t \to t + \pi$, is investigated. The structure of the modular double in the quantum case is introduced.

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1. Introduction

The Liouville model, based on the famous Liouville equation [1]

$$\varphi_{tt} - \varphi_{xx} + e^{2\varphi} = 0 \tag{1}$$

for the real function $\varphi(x, t)$ on the two-dimensional spacetime plays an important role in mathematics and physics. In the quantum variant it gives an example of conformal field theory and enters as a basic ingredient into Polyakov's theory of noncritical string [2]. In this application the spacetime is taken as a cylinder $\mathbb{R}^1 \times \mathbb{S}^1$, so that $\varphi(x, t)$ satisfies the periodicity condition

$$\varphi(x + 2\pi, t) = \varphi(x, t), \tag{2}$$

and it is especially important to describe the spectrum of primary fields or, in other words, zero modes. This picture was thoroughly investigated in the early 1980s by canonical quantization [3–5] and more recently by means of the theory of representations for the quantum group $\mathcal{U}_q(\mathrm{sl}(2, \mathbb{R}))$ in [6, 7]. In this paper we shall develop an alternative approach, based on the methods of old papers [5, 8]. The main result is the following observation: the zero modes can be combined into dynamical variables Z_n with a simple evolution equation, corresponding to the discrete shift $t \to t + \pi$, which looks as follows:

$$Z_{n+1} Z_{n-1} = 1 + Z_n^2 \tag{3}$$

(with the quantum correction introduced in the main text). Thus for such time shifts the zero modes completely decouple from the infinite number of the oscillator degrees of freedom.

The main interest is in the spectrum of the corresponding quantum Hamiltonian. We shall confirm the known result that there exists a quantization, for which this spectrum is simple, continuous and positive. However, we believe that our approach could lead to an alternative quantization, which is now under investigation.

Let us mention in passing that the system (3) gives a basic example in the modern theory of cluster algebras [9]. The symplectic approach of our paper could be instructive for the specialists in this topic.

The plan of the paper is as follows.

In section 1, we derive system (3) in a classical setting. In section 2, the zero-curvature representation of the Liouville model is recalled. The explicit solution of (3) and its canonical interpretation is given in section 3. Section 4 is devoted to the quantization. Finally, in section 5, we discuss a new feature, inherent only to the quantum case—the modular dual dynamical model. It is at this point where our hope for an alternative quantization lies.

2. Derivation of (3)

As is well known, the general solution of the Liouville equation (1) is given by the Liouville formula [1]

$$\mathrm{e}^{2\varphi(x,t)} = -4\frac{f'(x-t)g'(x+t)}{(f(x-t)-g(x+t))^2}, \tag{4}$$

where $f(x)$ and $g(x)$ are arbitrary real functions of one variable and f', g' are their derivatives. For a given $\varphi(x,t)$ we shall call f and g its representative functions. The correspondence

$$\varphi \leftrightarrow (f, g)$$

is not one to one. In particular φ is invariant with respect to a fractional-linear (Möbius) transformation

$$f \to \frac{af+c}{bf+d}, \qquad g \to \frac{ag+c}{bg+d} \tag{5}$$

for any real unimodular matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

For the RHS of (4) to be positive and finite it is enough to require that f and g are correspondingly monotonically decreasing and increasing,

$$f' < 0, \qquad g' > 0. \tag{6}$$

These conditions are compatible with transformation (5). We shall refer to (6) as the positivity condition.

The periodicity (2) can be achieved if the representative functions satisfy the relation

$$f(x+2\pi) = \frac{Af(x)+C}{Bf(x)+D}, \qquad g(x+2\pi) = \frac{Ag(x)+C}{Bg(x)+D}$$

with the monodromy matrix

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

depending on the solution $\varphi(x,t)$ and the choice of the representative functions f and g. The trace of monodromy

$$A + D = 2\cosh P$$

is representation invariant and defines an important dynamical variable P, which is called the quasimomentum of zero modes.

With these preliminaries we can turn to the derivation of (3). Take a shift Δ-positive number and construct a combination of the representative functions f and g which is a cross-ratio

$$Y(x,t)[\varphi] = -\frac{(f(x-t+\Delta) - f(x-t-\Delta))(g(x+t+\Delta) - g(x+t-\Delta))}{(f(x-t+\Delta) - g(x+t+\Delta))(f(x-t-\Delta) - g(x+t-\Delta))}.$$

This combination is positive due to the positivity condition. It follows from the Möbius invariance that $Y[\varphi]$ does not depend on the choice of the representative functions f and g. It is also clear that for $\Delta \to 0$,

$$Y(x,t)[\varphi] \sim \Delta^2\,\mathrm{e}^{2\varphi(x,t)}.$$

Now consider four values of $Y(x,t)[\varphi]$,

$$Y_N = Y(x,t+\Delta)[\varphi], \qquad Y_W = Y(x-\Delta,t)[\varphi],$$
$$Y_E = Y(x+\Delta,t)[\varphi], \qquad Y_S = Y(x,t-\Delta)[\varphi].$$

For fixed x,t these four cross-ratios depend on six numbers $f(x-t)$, $f(x-t\pm\Delta)$, $g(x+t)$, $g(x+t\pm\Delta)$ and due to the Möbius invariance only three of them are independent. Therefore, there exists a relation between them and indeed it is easy to check that this relation has the form

$$Y_N Y_S = \frac{Y_W Y_E}{(1+Y_W)(1+Y_E)}. \tag{7}$$

This consideration was used before in [10], where the discretized version of the Liouville equation was introduced. Also mention that (7) gives the simplest example of so-called Y-systems [11].

Now recall that $Y(x,t)[\varphi]$ is 2π-periodic in x. Hence, if $\Delta = \pi$, then $Y_W = Y_E$ and equation (7) becomes

$$Y_N Y_S = \frac{Y_W^2}{(1+Y_W)^2} = \frac{Y_E^2}{(1+Y_E)^2},$$

or in terms of inverse square roots $Z = 1/\sqrt{Y}$,

$$Z_N Z_S = 1 + Z_W^2 = 1 + Z_E^2.$$

Hence, clearly, the sequence

$$Z_n = \frac{1}{\sqrt{Y((n+1)\pi, n\pi)}}$$

satisfies the recurrence relation

$$Z_{n+1} Z_{n-1} = 1 + Z_n^2.$$

3. Zero curvature and the representative functions

We recall here the zero curvature of the Liouville model, introduced in [5]. The pair of Lax operators

$$L_1 = \frac{\mathrm{d}}{\mathrm{d}x} - L, \qquad L_0 = \frac{\mathrm{d}}{\mathrm{d}t} - M$$

with the traceless (2×2) matrices

$$L = \frac{1}{2}\begin{pmatrix} \varphi_t(x,t) & \mathrm{e}^{\varphi(x,t)} \\ \mathrm{e}^{\varphi(x,t)} & -\varphi_t(x,t) \end{pmatrix}; \tag{8}$$

$$M = \frac{1}{2}\begin{pmatrix} \varphi_x(x,t) & -e^{\varphi(x,t)} \\ e^{\varphi(x,t)} & -\varphi_x(x,t) \end{pmatrix}, \tag{9}$$

produces equation (1) as a zero-curvature condition

$$L_t - M_x + [L, M] = 0.$$

Mention that in contrast to the more generic case (see, e.g., [12]) this Lax pair does not contain a spectral parameter, so it corresponds to the finite-dimensional Lie algebra $\mathrm{sl}(2, \mathbb{R})$ rather than to its affine extension. In fact, one can get these Lax operators as a degenerate case of those for the sine-Gordon equation from [12].

Let

$$T(x,t) = \begin{pmatrix} A(x,t) & B(x,t) \\ C(x,t) & D(x,t) \end{pmatrix}$$

be a solution of the compatible equations

$$T_x = LT, \qquad T_t = MT \tag{10}$$

with the initial condition

$$T(0,0) = I.$$

This solution is a unimodular hyperbolic matrix

$$\det T = AD - BC = 1; \qquad \mathrm{tr}\, T = A + D > 2$$

and satisfies the quasiperiodycity condition

$$T(x + 2\pi, 0) = T(x, 0)T(2\pi, 0). \tag{11}$$

The representative functions $f(x)$ and $g(x)$ are easily expressed via matrix elements of $T(x,t)$. In more detail, consider the ratios

$$f(x,t) = \frac{A(x,t)}{B(x,t)}, \qquad g(x,t) = \frac{C(x,t)}{D(x,t)}. \tag{12}$$

It easily follows from (10) that

$$f_x = -f_t = -\frac{e^{\varphi}}{2B^2}, \qquad g_x = g_t = \frac{e^{\varphi}}{2D^2}.$$

Hence

$$f(x,t) = f(x-t), \qquad g(x,t) = g(x+t).$$

Furthermore, we have from (12)

$$f - g = \frac{AD - BC}{BD} = \frac{1}{BD}$$

so that

$$-4\frac{f'(x-t)g'(x+t)}{(f(x-t) - g(x-t))^2} = e^{2\varphi}.$$

Finally the quasiperiodicity (11) in terms of f and g looks as follows:

$$f(x+2\pi) = \frac{Af(x) + C}{Bf(x) + D}, \qquad g(x+2\pi) = \frac{Ag(x) + C}{Bg(x) + D}, \tag{13}$$

where A, B, C, D are matrix elements of

$$T = T(2\pi, 0) = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

so that $T(2\pi, 0)$ indeed plays the role of monodromy. Thus $f(x)$ and $g(x)$ give us a particular set of representative functions, normalized by the condition

$$f(0) = \infty, \qquad g(0) = 0. \tag{14}$$

Now we shall connect the initial data Z_{-1}, Z_0 for our system for Z_n with the elements of monodromy. We have

$$\begin{aligned} Z_{-1} &= (Y(0,-\pi))^{-1/2} = \left(-\frac{(f(2\pi)-f(0))(g(0)-g(-2\pi))}{(f(2\pi)-g(0))(f(0)-g(-2\pi))}\right)^{-1/2} \\ &= \left(-\frac{g(-2\pi)}{f(2\pi)}\right)^{-1/2} = \frac{A}{\sqrt{BC}}; \end{aligned} \tag{15}$$

$$\begin{aligned} Z_0 &= (Y(\pi,0))^{-1/2} = \left(-\frac{(f(2\pi)-f(0))(g(2\pi)-g(0))}{(f(2\pi)-g(2\pi))(f(0)-g(0))}\right)^{-1/2} \\ &= \left(\frac{g(2\pi)}{f(2\pi)-g(2\pi)}\right)^{-1/2} = \frac{1}{\sqrt{BC}}. \end{aligned} \tag{16}$$

We used here the normalization (14) and quasiperiodicity (11).

We conclude this section by expressing the monodromy via Z_{-1}, Z_0. From (15), (16) we have

$$A = \frac{Z_{-1}}{Z_0}, \qquad BC = \frac{1}{Z_o^2}.$$

Then

$$D = \frac{1+BC}{A} = \frac{1+Z_0^{-2}}{Z_{-1}} = \frac{1+Z_0^2}{Z_{-1}Z_0} = \frac{Z_0}{Z_{-1}} + \frac{1}{Z_{-1}Z_0}.$$

The only unknown remains the ratio $\alpha^2 = B/C$. In terms of Z_{-1}, Z_0 and α the monodromy looks as follows:

$$T = \begin{pmatrix} \frac{Z_{-1}}{Z_0} & \frac{\alpha}{Z_0} \\ \frac{1}{\alpha Z_0} & \frac{Z_0}{Z_{-1}} + \frac{1}{Z_0 Z_{-1}} \end{pmatrix}.$$

4. Explicit solution and its canonical interpretation

It is easily shown that the solution of our system for Z_n can be given rather explicitly via the parametrization

$$Z_n = \frac{\cosh(Q+nP)}{|\sinh P|}, \tag{17}$$

where Q and P are to be found from the initial data

$$Z_{-1} = \frac{\cosh(Q-P)}{|\sinh P|}, \qquad Z_0 = \frac{\cosh Q}{|\sinh P|}.$$

Thus the evolution $Z_n \to Z_{n+1}$ in new variables consists of the mere shift

$$P \to P, \qquad Q \to Q + P.$$

Variable P is connected with the invariant of the monodromy

$$A + D = \frac{1}{Z_{-1}Z_0} + \frac{Z_0}{Z_{-1}} + \frac{Z_{-1}}{Z_0} = 2\cosh P.$$

The expression for Q is less transparent and we shall not present it here. Its connection with the fixed points of monodromy is discussed in [5].

Let us give the canonical interpretation for these variables, returning first to the full Liouville model and following [3, 5]. Equation (1) has a natural canonical presentation with the phase-space coordinates being two functions on the circle, namely

$$\varphi(x) = \varphi(x,0), \qquad \pi(x) = \varphi_t(x,0)$$

with Poisson brackets

$$\{\pi(x), \varphi(x)\} = \gamma\delta(x-y) \tag{18}$$

and Hamiltonian

$$H = \frac{1}{2\gamma}\int_0^{2\pi}\left(\pi^2(x) + \varphi_x^2(x) + \mathrm{e}^{2\varphi(x)}\right)\mathrm{d}x.$$

The positive coupling constant γ does not enter the equation of motion, however we shall retain it for future quantization.

It follows from (18) that the elements of the Lax matrix L (8) have the Poisson relations, which can be written as

$$\{L(x) \overset{\otimes}{,} L(y)\} = [r, L(x)\otimes I + I \otimes L(x)]\delta(x-y), \tag{19}$$

if we use the general notations from [12]. Here r is a classical r-matrix (in fact historically the first example of such matrices)

$$r = \frac{\gamma}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The local relation (19) leads to the relation for the monodromy

$$\{T \overset{\otimes}{,} T\} = [r, T\otimes T]$$

or explicitly

$$\begin{aligned} &\{A, B\} = \frac{\gamma}{2}AB, && \{A, C\} = \frac{\gamma}{2}AC \\ &\{D, B\} = -\frac{\gamma}{2}BD, && \{D, C\} = -\frac{\gamma}{2}CD \\ &\{A, D\} = 2\gamma BC, && \{B, C\} = 0. \end{aligned}$$

This allows us to calculate brackets for the initial data using expressions (15) and (16). We get

$$\{Z_{-1}, Z_0\} = -\frac{\gamma}{2}Z_{-1}Z_0.$$

Finally, it follows from the relation

$$\mathrm{d}\ln Z_{-1}\wedge \mathrm{d}\ln Z_0 = -\mathrm{d}P\wedge\mathrm{d}Q$$

that P and Q have canonical brackets

$$\{P, Q\} = \frac{\gamma}{2}.$$

However the phase space for P , Q is not the full plane, but rather half of it. Indeed the map

$$(P, Q) \to (Z_{-1}, Z_0)$$

for positive Z_{-1}, Z_0 has as a pre-image one of the half-planes $P > 0$ or $P < 0$. Thus the phase space of P, Q is $\mathbb{R}^2/Z_2$. It is clearly inconvenient for the quantization.

The way out consists of the following: consider the parametrization

$$Z_n = \mathrm{e}^{Q+nP} + \frac{1}{4\sinh^2 P}\,\mathrm{e}^{-Q-nP}, \tag{20}$$

which differs from (17) by the shift of Q,

$$Q \to Q + \ln 2|\sinh P|,$$

which does not change the canonical structure. It is easy to see that such Z_n are invariant under the action of canonical transformation K,

$$K : P \to -P; \qquad Q \to -Q - \ln(4\sinh^2 P),$$

which is a superposition of a simple reflection

$$K_0 : P \to -P; \qquad Q \to -Q$$

and point transformation

$$S : P \to P; \qquad Q \to Q - \ln(4\sinh^2 P).$$

The latter is defined via the action

$$Q \to Q - \frac{\mathrm{d}S(P)}{\mathrm{d}P},$$

with

$$S(P) = \int^P \ln(4\sinh^2 P)\,\mathrm{d}P. \tag{21}$$

The last integral reduces to the dilogarithm

$$\int \ln\left(t - \frac{1}{t}\right)^2 \frac{\mathrm{d}t}{t} = 2\int (\ln(t-1) + \ln(t+1) - \ln t)\frac{\mathrm{d}t}{t}.$$

It is clear that K is a reflection

$$K \circ K = \mathrm{id}$$

which interchanges the half-planes. Its connection with the Weyl reflection for the monodromy is evident.

All this allows us to consider the variables Z_n on the whole plane $\mathbb{R}^2$ and approach to quantization correspondingly, taking into account their invariance under reflection.

5. Quantization

Formula (20) gives Z_n as a rational function of exponents e^P and e^Q with the relation

$$\{\mathrm{e}^P, \mathrm{e}^Q\} = \gamma \mathrm{e}^P\,\mathrm{e}^Q.$$

Under the canonical quantization

$$\{P, Q\} \to \frac{\mathrm{i}}{\hbar}[P, Q]$$

these exponents turn into the Weyl operators u, v with the commutation relation

$$uv = q^{-1}vu, \qquad q = \mathrm{e}^{\mathrm{i}\gamma/2}$$

(we put $\hbar = 1$). Naive ordering of factors in (20) leads to the formula

$$Z_n = q^{-n/2}vu^n + q^{n/2}\frac{1}{u-u^{-1}}u^{-n}v^{-1}\frac{1}{u-u^{-1}}. \tag{22}$$

It is easy to check that

$$Z_{n-1}Z_n = qZ_nZ_{n-1}$$

and

$$Z_{n+1}Z_{n-1} = 1 + q^{-1}Z_n^2.$$

Thus formula (22) defines a quantization of the system (3). If operators u, v are self-adjoint and positive, the same is true for Z_n, if $\bar{q} = q^{-1}$, which is valid for real γ.

Unitary operator U, such that

$$U^{-1}uU = u; \qquad U^{-1}vU = q^{-1/2}vu,$$

realized also the evolution

$$Z_{n+1} = U^{-1}Z_nU. \tag{23}$$

The Hamiltonian H is given by

$$U = \mathrm{e}^{-2\pi\mathrm{i}H}. \tag{24}$$

The Weyl operators u and v are usually realized via multiplication and shift. We shall use the explicit formulae with a particular parametrization of the coupling constant γ. Its convenience will become clear later. First we renormalize γ putting

$$\gamma = 2\pi\tau,$$

so that the phase factor $q = \mathrm{e}^{\mathrm{i}\gamma/2}$ becomes $q = \mathrm{e}^{\mathrm{i}\pi\tau}$, reminiscenting an object of the automorphic functions theory. Following the tradition, stemming from Weierstrass, we introduce two complex parameters ω, ω' with the normalization

$$\omega\omega' = -\tfrac{1}{4}$$

and put

$$\tau = \frac{\omega'}{\omega}.$$

For τ to be real and positive, we take ω, ω' as pure imaginary with the positive imaginary part

$$\overline{\omega} = -\omega, \qquad \overline{\omega}' = -\omega'.$$

Operators u and v act on the Hilbert space $L_2(\mathbb{R})$ with elements $\psi(s)$, $- < s < \infty$, and the scalar product

$$(\psi', \psi) = \int_{-\infty}^{\infty} \overline{\psi}'(s)\psi(s)\,\mathrm{d}s$$

as follows,

$$u\psi(s) = \mathrm{e}^{\mathrm{i}\pi s/\omega}\psi(s), \qquad v\psi(s) = \psi(s+\omega'). \tag{25}$$

These operators are unbounded; an admissible domain of definition D consists of function $\psi(s)$, analytic in the whole complex plane $\mathbb{C}$ and rapidly vanishing along contours, paralleled to the real axis. Functions of the form

$$\psi(s) = \mathrm{e}^{-s^2}P(s),$$

where $P(s)$-polynomial is a representative for D. With this definition operators u and v are essentially self-adjoint and positive definite. The same is true for Z_n.

The Hilbert space $L_2(\mathbb{R})$ realizes the quantization of the phase space, which is the whole plane $\mathbb{R}^2$. The reduction to the half-plane after quantization should use the quantum analogue of the canonical transformation K. Let us construct the corresponding unitary operator such that

$$KZ_n = Z_n K, \qquad K^2 = I.$$

As in the classical case we put

$$K = K_0 S,$$

where K_0 is a simple reflection

$$K_0^{-1} u K_0 = u^{-1}, \qquad K_0^{-1} v K_0 = v^{-1}$$

and S realizes the transformation

$$S^{-1} u S = u; \qquad S^{-1} v S = (u - u^{-1}) v (u - u^{-1}). \tag{26}$$

Due to commutativity with u, S is a multiplication operator

$$S\psi(s) = S(s)\psi(s).$$

From the quasiclassical consideration S must be expressed via the exponent of the deformed classical action from (21), which involves dilogarithm. The candidate for this is already known, see [13, 14]. It is a meromorphic function $\gamma(\xi)$ defined via functional equations

$$\frac{\gamma(\xi + \omega')}{\gamma(\xi - \omega')} = 1 + \mathrm{e}^{-\mathrm{i}\pi\xi/\omega}, \tag{27}$$

having properties

$$\gamma(\xi)\gamma(-\xi) = \mathrm{e}^{\mathrm{i}\pi\xi^2}, \qquad \overline{\gamma(\xi)} = \frac{1}{\gamma(\bar{\xi})}. \tag{28}$$

The proper definition and beautiful properties of this function, which generalizes both exponents, dilogarithm and Γ-function, are described in [15].

Let M be a multiplication operator by the function

$$M(s) = \mathrm{e}^{-2\pi\mathrm{i}s(s+\omega'')}\gamma(2s + \omega''),$$

where

$$\omega'' = \omega + \omega'.$$

It follows from (28) that for real s,

$$M(-s) = \overline{M}(s)$$

and we get from (27)

$$\frac{M(s+\omega')}{M(s)} = \mathrm{i}\big(q\,\mathrm{e}^{\frac{\mathrm{i}\pi s}{w}} - q^{-1}\mathrm{e}^{-\frac{\mathrm{i}\pi s}{w}}\big).$$

Thus

$$M^{-1} v M = \frac{M(s+\omega')}{M(s)} v = \mathrm{i}(qu - q^{-1}u^{-1})v = \mathrm{i}v(u - u^{-1}).$$

Now we put

$$S(s) = \frac{M(s)}{M(-s)}$$

and finally get

$$S^{-1} v S = (u - u^{-1}) v (u - u^{-1})$$

as was required in (26). Operator S is unitary

$$\overline{S(s)} = \frac{\overline{M(s)}}{\overline{M(-s)}} = \frac{M(-s)}{M(s)} = S^{-1}(s) = S(-s)$$

so that

$$K^2 = (K_0 S)^2 = K_0 S K_0 S = S^{-1} S = I;$$

thus K is indeed a reflection.

We can introduce a pair of projectors

$$\Pi_\pm = \tfrac{1}{2}(I \pm K),$$

which commute with the operators Z_n. Thus each of the reduced Hilbert spaces

$$\mathcal{H}_\pm = \Pi_\pm L_2(\mathbb{R})$$

can be considered as a physical Hilbet space in which operators Z_n are defined. This reduction is the quantum analogue of using half-plane as a classical phase space.

The evolution (23) is given by

$$U\psi(s) = \mathrm{e}^{-2\pi \mathrm{i} H}\psi(s) = \mathrm{e}^{-2\pi \mathrm{i} s^2}\psi(s),$$

in other words, H is the operator of multiplication by s^2. Its spectrum is continuous and positive. In the whole $L_2(\mathbb{R})$ it has multiplicity 2 with the generalized eigenfunctions $\delta(s+k)$ and $\delta(s-k)$. However, in the reduced space only one combination is an eigenfunction; for example in the space $\mathcal{H}_+$ it is

$$\delta(s-a) + S(k)\delta(s+a).$$

So the physical spectrum of H is simple. We cannot help mentioning the analogy of the last formula with those of the scattering theory involving the incoming and outgoing waves, connected by the S-matrix. In fact, this resemblance is not accidental. We hope to return to it, discussing a natural quantum deformation of the Gindikin–Karpelevich formula in the future.

6. Modular double

As was already mentioned some time before [16], the Weyl system (25), acting in $L_2(\mathbb{R})$, has a dual partner

$$\tilde{u}\psi(s) = \mathrm{e}^{\mathrm{i}\pi s/\omega'}\psi(s), \qquad \tilde{v}\psi(s) = \psi(s+\omega)$$

with relations

$$\tilde{u}\tilde{v} = \tilde{q}^{-1}\tilde{v}\tilde{u}, \qquad \tilde{q} = \mathrm{e}^{\mathrm{i}\pi/\tau}.$$

It is here where we see the convenience of our variables ω, ω': dualization corresponds to the interchange $\omega \leftrightarrow \omega'$.

Operators $\tilde{u}, \tilde{v}$ are also positive self-adjoint for real positive τ. They have a simple commutation relation with u, v,

$$u\tilde{u} = \tilde{u}u, \qquad v\tilde{v} = \tilde{v}v; \qquad u\tilde{v} = -\tilde{v}u, \qquad v\tilde{u} = -\tilde{u}v.$$

We can use them to construct dual variables $\tilde{Z}_n$ by formula (22) with substitution of u, v by the dual ones $\tilde{u}, \tilde{v}$. It is clear that operators Z_n and $\tilde{Z}_n$ commute. The operator K and Hamiltonian H are invariant under interchange $\omega \leftrightarrow \omega'$ and so play for $\tilde{Z}_n$ the same role as for Z_n.

Thus we observe that after quantization we acquire a second dynamical system acting in the same Hilbert space. For a positive coupling constant the second system is independent of the first, both of them are self-consistent. However, our construction allows us to use another regime for τ, which is needed for the application of Liouville model to a noncritical string. Indeed, as is well known, the central charge C of the quantum Liouville model is given by the formula

$$C = 1 + \left(\tau + \frac{1}{\tau} + 2\right)$$

and it is a real number for real $\tau + \frac{1}{\tau}$, so that in addition to positive τ we can use τ such that

$$\overline{\tau} = \frac{1}{\tau}, \qquad |\tau| = 1. \tag{29}$$

For real positive τ we have $C > 25$, but for τ from (29)

$$1 < C < 25.$$

This is exactly this regime which is relevant to the noncritical string and we can use our doubling to treat this case.

Of course, the complex coupling constant raises the question of self-adjointness and unitary. However, in regime (29), which is described by the condition

$$\overline{\omega} = -\omega',$$

we have the involution

$$\widetilde{Z}_n^* = Z_n, \tag{30}$$

which interchanges our commuting systems. Abstractly this involution is similar to

$$(a \otimes b)^* = b^* \otimes a^*$$

in the tensor products. Thus we can consider (30) as a legitimate variant of the reality condition, valid in regime (29). Hamiltonian (24) can be used here as well, so it is the same as for the case of positive τ.

This apparently supports the conviction of the papers [6, 7, 17, 18] that smooth continuation of characteristics of the Liouville model, such as N-point correlation functions, calculated for $C > 25$, through the threshold $C = 25$ is the sole possibility. However, we still hope that one can find an alternative quantization of Z_n, $\widetilde{Z}_n$ in the regime (29), nonequivalent to the described here. This could vindicate our approach as not purely methodological.

One reason for optimism is the recent result of one of the authors (LDF) about the discrete series of representation for algebra $\mathcal{U}_q(\mathrm{sl}(2))$ [19] in regime (29). Work in this direction will be continued.

7. Conclusions

In this paper, we separated a finite-dimensional dynamical system for the zero modes of the Liouville model in its classical and quantized formulation. The new feature is the modular duality in the quantum case suitable to treat the case of strong coupling. Let us add one more statement.

In [10], the lattice variant of the Liouville model was proposed. The considerations of this paper could be straightforwardly applied to this case also. The equations for the zero modes are exactly the same as in this paper. We believe that this is another example of separation of degrees of freedom in CFT both in the continuous and lattice variants with zero modes not depending on lattice regularization. In a less explicit form this feature was already observed in [20] for the Wess–Zumino model.

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